ON THE STABILITY OF THE EQUILIBRIUM POSITIONS FOR DISCONTINUOUS SYSTEMS

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Consider two systems of differential equations

$$\frac{dx}{dt} = f^{\pm}(x) \qquad (1\pm) \qquad (1)$$

which represent a motion in phase space on the "upper side" (+) and on the "lower side"(−), respectively, of a given surface.*

$$F(x) = 0 \qquad (2)$$

Here x denotes an n-dimensional vector with coordinates $x_1, \ldots, x_n$; the functions $f^{\pm}(x)$ are vector-valued, and the function $F(x)$ is scalar-valued.

It is supposed that the "upper" (1^+), as well as the "lower" (1^-), system of equations satisfies the usual conditions which guarantee existence and uniqueness of solutions fulfilling given initial conditions, and do not have singular points on the discontinuity surface. Indeed, the systems ($1^{\pm}$) do not determine the transition conditions across the discontinuity surface or the motion along it, which remains to be determined. Thus, on the surface of discontinuity there may appear positions of equilibrium.

In certain cases the question of the stability of these positions of equilibrium can be ascertained in a simple manner, starting from the new equations which have to be introduced to determine the problem and to detect the occurrence of equilibrium. In other, much more difficult,

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* In this paper, for brevity, we employ the terms "surface" and "plane" instead of the usual terms "hypersurface" and "hyperplane" in n-dimensional space, respectively.

cases, in spite of equilibrium being governed by new equations, the equation of stability should be solved in principle from Equations (1).

The present paper is devoted to the discussion of the stability of equilibrium positions which occur on the surface of discontinuity.

This question, for second-order systems, was considered in [1]. But the fundamental case, which will interest us here later, when the vector fields $f^{\pm}$ are tangent to the surface of discontinuity at the point under consideration, was not investigated in [1]. In the case of arbitrary order n, similar questions have been considered only for relay systems [2,3,4,5]

$$\frac{dx}{dt} = Ax + \lambda \frac{x_1}{|x_1|} \tag{3}$$

where A is a constant square matrix and λ is a constant vector, i.e. for systems which differ from the linear constant-coefficient systems only by a single nonlinear function of the relay type. We consider here the problem for differential equations of the type ($1^{\pm}$) with arbitrary right-hand sides.

1. Determination of the motion in phase space. The systems of equations ($1^{\pm}$) determine two vector fields on the surface of discontinuity. Consider one of them, for example the "upper" one, corresponding to the system (1^{+}). The surface of discontinuity is divided into domains, on each of which the vectors of the upper field are directed toward a definite side of the surface ("into" the surface of "away from" it). These domains are separated by an n-2 dimensional manifold Γ^{+}. Along points of Γ^{+} the vector field is tangent* to the surface.

Analogously, on the other side of the surface of discontinuity there are domains which are separated by a manifold Γ^{-}.

Each point of the manifold Γ^{+} (and analogously of Γ^{-}) is of one of two types, type A or type B (Fig. 1), depending upon the behavior of the phase trajectories of the system (1^{+}) (correspondingly, of the system (1^{-})) in the neighborhood of the point.

The manifolds Γ^{+} and Γ^{-} are divided into a "domain of gliding" C where the vectors of the upper and lower fields are directed opposite to each other ("collide", see Fig. 2a) and a regular part P where the

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* It is supposed that the phase trajectories of the systems (1^{+}) and (1^{-}) may be tangent to the surface of discontinuity (2) only at isolated points, these points of tangency being simple (*not* multiple).

vectors do not cancel each other out (or "collide")* (Figs. 2b and 2c).

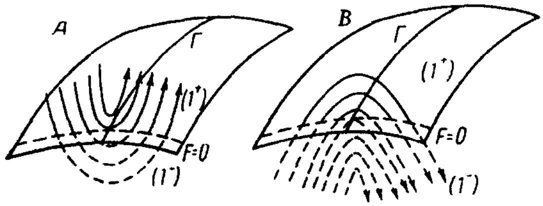

FIG. 1.

The motion of a representative point (or "particle") lying on the surface of discontinuity, proceeds as follows:

1. The trajectories are always continuous.

2. A particle which arrives at a point of the regular domain P of the surface of discontinuity from a half-space continues its motion into another half-space, moving along the trajectory which passes through the regular point † of the surface (Fig. 2b).

3. A particle on the domain of gliding C cannot continue its motion according to the systems $(1^{\pm})$ and must proceed along C on the surface of discontinuity. In addition, the differential equations which specify this motion must be given. We shall call these the equations § (C).

It is assumed that the singular points of the system (C) do not lie on the boundary of the domain C.

4. Suppose now that a particle passes through or is initially located at a point belonging to either of the manifolds Γ^+ or Γ^-. These manifolds may be subdivided into n-2 dimensional domains P_Γ and C_Γ. From

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* Inside the domain C there may be points at which the vector field is tangent to the surface, but such points are not of interest to us.

† In the case of Fig. 2b the particle, in general, does not proceed along the surface of discontinuity. If it is disturbed slightly at the initial moment, then the choice of the possible trajectories will be determined.

§ Equations (C), generally speaking, are not related in any way to equations $(1^{\pm})$. But, in a number of important special cases, the system (C) may be naturally defined in terms of $(1^{\pm})$. For more details concerning this, see [3,4,6,7].

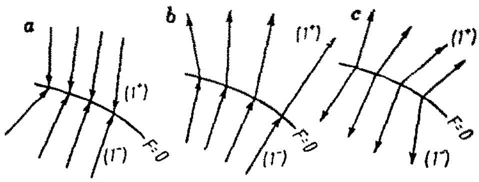

FIG. 2.

each point of P_Γ there issues at least one trajectory of the systems (1^+), (1^-), (C), while no trajectories issue out of points of C_Γ. When the particle lies in the domain P_Γ, then it departs along one of the trajectories*. If the particle is located on the domain C_Γ, then it proceeds further into the manifold C_Γ in accordance with the system of equations (Γ), which must be specified in addition.

The particle may be located in the n-3 dimensional manifold G, the intersection of the domains P_Γ and C_Γ. But this manifold G may likewise be subdivided into domains** P_G and C_G, and for C_G one must give, in addition, equations (C_G) and so on.

5. If the manifolds Γ^+ and Γ^- coincide, then the motion is determined as in point 4 above.

The case when the manifolds Γ^+ and Γ^- have a lower dimensional intersection, is not considered in this paper. Hence the further motion of a particle in this situation is left out of account.

The equilibrium positions on the discontinuity surface may occur only at points of the domain C and the manifold Γ which are singular for the (additional) systems of equations (C), (Γ), (G), and so forth.

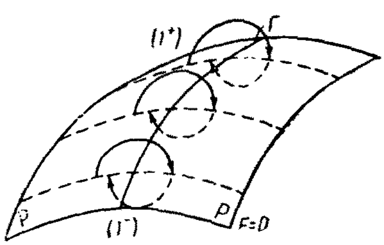

FIG. 3.

* If there are several such trajectories, then the choice of a trajectory must be specified.

** Out of every point of P there issues at least one trajectory of each of the systems $(1+)$, $(1-)$, (C), (Γ).

Let us look at a particular case which will play an important role in what follows. In this case (Fig. 3) the manifolds Γ^+ and Γ^- coincide, the point under consideration (position of equilibrium) belongs to the intersection of these manifolds, and both manifolds belong to type B (Fig. 1). In particular, on the discontinuity surface, on both sides of the coincident manifolds Γ, regular domains are situated.

If, in this case, the equilibrium position is a stable singular point of the system of equations (Γ), then there arises the question of the stability of the position of equilibrium with respect to initial disturbances, chosen out of an arbitrary n-dimensional neighborhood of the positions of equilibrium. In order to answer this question it is necessary to study the behavior, in such a neighborhood, of the phase trajectories of the system $(1^{\pm})$.

In all other (nonsingular) cases the trajectories of the system $(1^{\pm})$ guarantee the motion of the particle in a small neighborhood or on the discontinuity surface, and the question of stability may be answered in an elementary way, provided that the stability or instability of the singular points of the systems (C), (Γ), (G), etc. is known. Hence, we shall deal only with the stability in the singular case.

2. Formulas for point transformations in the singular case. The point under consideration will be supposed to be at the origin of coordinates, and we shall assume that the functions $f^+(x)$, $f^-(x)$, and $F(x)$ are either analytic or, in any case, sufficiently smooth* in a small neighborhood of the origin of coordinates. We are interested in the behavior of the integral curves only in the neighborhood of the origin and, therefore, when we speak from now on of space, surface, plane, half-plane, and so forth, we shall mean by this their intersection with a sufficiently small neighborhood of the origin.

Without loss of generality we may assume that the surface of discontinuity is the plane $x_n = 0$ and that the n-2 dimensional manifold Γ, consisting of the points at which the upper and the lower vector fields are simultaneously tangent to the surface of discontinuity, is defined

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* Each of the functions $f^+(x)$ and $f^-(x)$ is defined, not on a whole neighborhood of the point $x = 0$, but only on one of the portions into which such a neighborhood is split by the surface of discontinuity. When we speak of the analyticity and of the sufficient smoothness of these functions, we shall have in mind the possibility of extending the definition of these functions to a whole neighborhood of the point $x = 0$.

by the equations* $x_1 = 0$, $x_n = 0$; the system (1^+) acts for $x_n > 0$, and (1^-) acts for $x_n < 0$.

The integral trajectories of the system (1^+) determine a point transformation G_1 of the half-plane $x_1 \geqslant 0$, $x_n = 0$ into the half-plane $x_1 \leqslant 0$, $x_n = 0$, and the trajectories of the system (1^-) define a point transformation G_2 of the half-plane $x_1 < 0$, $x_n = 0$ into the half-plane $x_1 \geqslant 0$, $x_n = 0$. A particle, proceeding along the integral curves of systems (1^+) and (1^-), traverses arc lengths s_1 and s_2 in times τ_1 and τ_2. The quantities s_1, s_2, τ_1 and τ_2 are functions of the initial point x on the plane of discontinuity.

The product of these two point transformations, $G = G_1G_2$, transforms the half-plane $x_1 > 0$, $x_n = 0$ into itself. All points of the manifold Γ are fixed points of the three transformations G_1, G_2, G. The stability of the fixed point $x = 0$, of the transformation G of the half-plane $x_1 > 0$, $x_n = 0$ into itself, is equivalent to the stability of an equilibrium position situated at the origin of coordinates** [8,9].

Consider the expansion of the components of the field $f^+(x)$ in the neighborhood of the origin of coordinates

$$f_j^+(x_1, \ldots, x_n) = c_j^+ + \sum_{k=1}^{n} c_{jk}^+ x_k + \ldots \qquad (j = 1, \ldots, n)$$

In view of the fact that at all points of the manifold Γ the field vectors lie in the plane $x_n = 0$, the function $f_n^+(x_1, \ldots, x_n)$ must vanish for $x_1 = 0$, $x_n = 0$.

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* The general case, when the surface of discontinuity is given by the equation $x_n = P(x_1, \ldots, x_{n-1})$ and the manifold Γ by the additional equation $x_1 = Q(x_2, \ldots, x_{n-1})$, may be reduced to the case actually considered here by means of the transformation of variables: $x_1' = x_1 - Q(x_2, \ldots, x_{n-1})$, $x_2' = x_2$, $\ldots$, $x_{n-1}' = x_{n-1}$, $x_n' = x_n - P(x_1, \ldots, x_{n-1})$.

** In the paper [9] this assertion is proved for the stability of periodic motions. This proof will carry over to the case of the stability of equilibrium positions, if it is taken into account that the time $\tau = \tau_1 + \tau_2$ approaches zero as x approaches the origin O. This is a consequence of the fact that the particle traverses the arcs s_1 and s_2 during the times τ_1 and τ_2, times which approach zero as $x \to 0$, and that the speed of traversal of these arcs approaches the lengths of the vectors $f^+(0)$ and $f^-(0)$, which are finite and not zero, since, by assumption, the origin of coordinates is not a singular point of the system (1^+) and (1^-).

Hence

$$f_n^+(x_1, \ldots, x_n) = c_{n1}^+ x_1 + c_{nn}^+ x_n + \ldots \tag{2.1}$$

Then, for the integral curve of the system (1^+) which passes through the origin of coordinates (this curve "cuts" the plane $x_n = 0$, since it has a contact of type B there) the dependence of the coordinate x_n on the time has the form

$$x_n = \frac{1}{2} \dot{f}_n^+(0) t^2 + \ldots \tag{2.2}$$

and, as is easily seen from (2.1), one has that $\dot{f}^+(0) = c_1^+ c_{n1}^+$. From the fact that this curve lies only on one side of the plane $x_n = 0$ it follows that the expansion begins with an even power of t. In the following we shall consider only the fundamental case* when this expansion begins with terms of order t^2, i.e. when $c_1^+ \neq 0$ and $c_{n1}^+ \neq 0$.

In order to obtain formulas for the transformation G_1, let us write the equations of the integral curves of the system (1^+) which issue from the point $(x_1, \ldots, x_{n-1}, 0)$:

$$y_j = x_j + t f_j^+(x_1, \ldots, x_{n-1}, 0) + \frac{1}{2} t^2 \dot{f}_j^+(x_1, \ldots, x_{n-1}, 0) + \ldots \quad (j = 1, \ldots, n) \tag{2.3}$$

and let us put, in the last equation, (for $j = n$), $y_n = x_n = 0$.

From the resulting equation, employing in an essential way the conditions $c_1^+ \neq 0$, $c_{n1}^+ \neq 0$, we obtain the time $t = \tau_1$ in the form of a series in $x_1, \ldots, x_{n-1}$. Putting this series in place of t in the remaining equations (2.3), we obtain the formulas for the transformation G_1.

Since all points of the manifold Γ ($x_1 = 0$) are fixed points of the mapping G_1, the formulas defining the transformation G_1 have the following form:

$$y_1 = p_1 x_1 + x_1 \varphi_1(x), \qquad y_j = b_j x_1 + x_j + x_1 \varphi_j(x) \qquad (j = 2, \ldots, n-1) \tag{2.4}$$

where the functions $\phi_1(x), \ldots, \phi_{n-1}(x)$ have power series expansions which begin with linear terms.

The above process leads to the formulas

$$b_j = -2 \frac{c_j^+}{c_1^+} \qquad (j = 2, \ldots, n-1) \tag{2.5}$$

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* See footnote, p. 407. In the case of a contact of type B, one has that $c_1^+ c_{n1}^+ < 0$. Since in the case under consideration $c_1^+ < 0$, it follows that $c_{n1}^+ > 0$.

Formulas (2.4) were obtained under the assumption that after a time τ_1 the integral curves transform a point of the half-plane $x_1 > 0$, $x_n = 0$ into a point with $x_1 \leqslant 0$, $x_n = 0$. On the other hand, these formulas may be applied also to points on the half-plane $x_1 < 0$, and then for negative values of τ_1 the integral curves lead back to the initial points on the half-plane $x_1 > 0$, $x_n = 0$. In this sense the transformation (2.4) may be considered as a transformation of the whole plane into itself (see Section 2) involving only the integral curves of the system (1^+). This transformation is an involution, i.e. it coincides with its own inverse transformation. Therefore

$$x_1 = p_1 y_1 + y_1 \varphi_1(y), \qquad x_j = b_j y_1 + y_j + y_1 \varphi_j(y) \qquad (j = 2, \ldots, n-1) \qquad (2.6)$$

Substituting from the first of the equations (2.4) into the right-hand side of the first of the equations (2.6), we obtain that $p_1^2 = 1$; consequently $p_1 = -1$. (The possibility $p_1 = +1$ is excluded, since G_1 transforms the half-plane $x_1 > 0$ into the half-plane $x_1 < 0$.)

In particular, $p_1 = -1$ implies that $x_1 \phi_1(x) = y_1 \phi_1(y)$. Taking into account the first equation of (2.4) and eliminating x_1, we obtain $\phi_1(x) = [\phi_1(x) - 1]\phi_1(y)$. Putting here $x_1 = 0$, so that then $x = y$, it follows that

$$2\varphi_1(x) = [\varphi_1(x)]^2 \qquad \text{for } x_1 = 0$$

The expansion of $\phi_1(x)$ does not have a constant term; hence $\phi_1(0) = 0$ for $x_1 = 0$. But then $\phi_1(x) = x_1 \psi(x)$ and Formulas (2.4) for G_1 become

$$y_1 = -x_1 + x_1^2 \psi(x), \qquad y_j = b_j x_j + x_1 \varphi_j(x) \qquad (j = 2, \ldots, n-1) \qquad (2.7)$$

where the series expansions in x of the functions $\psi(x)$, contrary to what happens to the functions $\phi_j(x)$, may begin with a constant term.

In an entirely analogous manner, the point transformation G_2 is found to be given by the formulas

$$x_1^{(1)} = -y_1 + y_1^2 \psi^\circ(y), \qquad x_j^{(1)} = b_j^\circ y_1 + y_j + y_1 \varphi_j^\circ(y) \qquad (j = 2, \ldots, n-1) \qquad (2.8)$$

From (2.7) and (2.8) we obtain the following formulas for the transformation $G = G_1 G_2$:

$$x_1^{(1)} = x_1 + x_1^2 g(x), \qquad x_j^{(1)} = S_j x_1 + x_j + x_1 g_j(x) \qquad (j = 2, \ldots, n-1) \qquad (2.9)$$

where

$$S_j = b_j - b_j^\circ \qquad (j = 2, \ldots, n-1) \qquad (2.10)$$

$$g(x) = -\psi(x) + [x_1 \psi(x) - 1]^2 \psi(y) \qquad (2.11)$$

$$g_j(x) = \varphi_j(x) + b_j^\circ x_1 \psi(x) + [x_1 \psi(x) - 1]\varphi_j^\circ(y) \qquad (j = 2, \ldots, n-1)$$

Here, y is given as function of x by Equations (2.7). The expansions of the functions $g_1(x), \ldots, g_{n-1}(x)$ begins with linear terms, and $g(0)$ may differ from zero. Employing (2.5) and an analogous formula for b_j^0, Equation (2.10) yields

$$S_j = -2\left(\frac{c_j^+}{c_1^+} - \frac{c_j^-}{c_1^-}\right) \qquad (j = 2, \ldots, n-1) \tag{2.12}$$

It should be noticed that the point transformation of the half-plane $x_1 > 0$, $x_n = 0$ into itself which has been obtained in (2.9), involves the critical case, since all roots of the characteristic equation relative to the linear part of the transformation G are equal to unity.

3. Discussion of the stability in the singular case. Let us prove that the position of equilibrium $x = 0$ is unstable when the vectors $f^+(0)$ and $f^-(0)$ are non-colinear (i.e. not all $S_j = 0$). If all these vectors are colinear (all $S_j = 0$) then we shall give both a stability and an instability condition.

Lemma. If the fixed point $x = 0$ of the point transformation of the half-space $x_1 > 0$ into itself is stable, and if the first of the equations which define this transformation has the form

$$x_1^{(1)} = x_1 + x_1^2 g(x) \tag{3.1}$$

then one has for the iterations

$$x_1^{(0)} + x_1^{(1)} + x_1^{(2)} + \ldots = \infty \qquad (x_1^{(0)} = x_1 > 0) \tag{3.2}$$

*Proof.** Let us choose a sufficiently large number $M > 0$ such that the following inequalities hold:

$$|g(x)| < M, \quad |x^{(k)}| < \frac{1}{2M} \qquad (k = 0, 1, 2, \ldots) \tag{3.3}$$

whenever $|x| < \Delta$, $x_1 > 0$. Then, from (3.1) and (3.3), for these values of x we have

$$x_1^{(1)} > x_1 - Mx_1^2 = x_1(1 - Mx_1) > 0 \tag{3.4}$$

Let us consider the auxiliary scalar transformation

$$\alpha_1 = \alpha_0(1 - M\alpha_0) \tag{3.5}$$

and its sequence of iterations

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* In the proof of the lemma we employ a reasoning which was used by Levi-Civita [10] in discussing the two-sided stability (as $t \to \pm\infty$) of fixed points of transformations of the two-dimensional plane into itself.

$$\alpha_{m+1} = \alpha_m (1 - M\alpha_m) \qquad (m = 1, 2, \ldots) \tag{3.6}$$

Then, setting $\alpha_0 = x_1 > 0$ and comparing (3.5) with (3.4) we obtain that $x_1^{(1)} > \alpha_1 > 0$, and since

$$\frac{d}{dx_1}(x_1^2 - Mx_1) > 0 \quad \text{for } 0 < x_1 < \frac{1}{2M}$$

it follows that $x_1^{(2)} > x_1^{(1)} - Mx_1^{(1)^2} > \alpha_1 - M\alpha_1^2 = \alpha_2 > 0$ and, in general, that

$$x_1^{(m)} > \alpha_m > 0 \qquad (m = 1, 2, \ldots) \tag{3.7}$$

However, the following limit exists and is finite

$$l = \lim_{m\to\infty} \alpha_m = \lim_{m\to\infty} \alpha_0 \prod_{k=0}^{m-1} (1 - M\alpha_k) = x_1 \prod_{k=0}^{\infty} (1 - M\alpha_k)$$

From (3.6), letting $m \to \infty$, it follows that $l = l(1 - Ml)$, i.e. $l = 0$. As is well known, if

$$\prod_{k=0}^{\infty} (1 - M\alpha_k) = 0$$

then $\alpha_0 + \alpha_1 + \alpha_2 + \ldots = \infty$. From this fact and from (3.7), we conclude that $x_1^{(0)} + x_1^{(1)} + x_1^{(2)} + \ldots = \infty$, and the proof of the lemma is complete.

Theorem 1. If, in the particular case, at the position of equilibrium $x = 0$ the right-hand sides of the equations (1^+) and (1^-), i.e. the vectors $f^+(0)$ and $f^-(0)$, are not colinear, then this position of equilibrium is stable.

Proof. In view of the non-colinearity of the vectors $f^+(0)$ and $f^-(0)$, in Formulas (2.9) at least one, the S_j, is not zero. Suppose, for definiteness, that $S_2 \neq 0$. Without loss of generality we may suppose that $S_2 > 0$, since the case $S_2 < 0$ may be reduced to this merely by reversing the direction of the x_2 axis.

From (2.9) it follows that

$$x_2^{(1)} > x_2 + \tfrac{1}{2} S_2 x_1 \tag{3.8}$$

whenever $|x| < \Delta$ for sufficiently small $\Delta > 0$.

Let us suppose that the position of equilibrium under consideration is stable. Then the fixed point $x = 0$ of the transformation G which maps the n-1 dimensional half-space $x_1 \geqslant 0$, $x_n = 0$ into itself is stable.

This means that, for some $\delta = \delta(\Delta) > 0$, the inequalities $x_1 \geqslant 0$, $|x| < \delta$ it follows that

$$|x^{(k)}| < \Delta, \qquad x_1^{(k)} \geqslant 0 \qquad (k = 0, 1, 2, \ldots)$$

Thus, iterating the inequality (3.8), we obtain

$$\Delta > x_2^{(m)} > x_2 + \tfrac{1}{2} S_2 \sum_{k=0}^{m-1} x_1^{(k)} \qquad (m = 0, 1, \ldots)$$

But this implies that $x_1^{(0)} + x_1^{(1)} + x_1^{(2)} + \ldots < \infty$, contradicting the lemma. Hence $x = 0$ is an unstable fixed point, and the theorem is proved.

Let us now suppose that all

$$S_j = 0 \qquad (j = 2, \ldots, n-1) \tag{3.9}$$

i.e. the vectors $f^+(0)$ and $f^-(0)$ are colinear. In this case Formulas (2.9) may be written in matrix form

$$x^{(1)} = x + x_1(Ax + z) \qquad (x_1 \geqslant 0) \tag{3.10}$$

where

$$A = \begin{Vmatrix} a_{11} & 0 & \ldots 0 \\ a_{21} & a_{22} & \ldots a_{2p} \\ \ldots & \ldots & \ldots \\ a_{p1} & a_{p2} & \ldots a_{pp} \end{Vmatrix} \qquad (a_{11} = g(0)) \tag{3.11}$$

and where z is a sum of nonlinear terms, and $p = n - 1$. Thus, we have the following theorems*:

Theorem 2. The fixed point $x = 0$ of the mapping (of the half-space $x_1 \geqslant 0$ into itself) given by (3.10) and the corresponding position of equilibrium $x = 0$ are stable whenever all eigenvalues of the matrix A possess negative real parts. The equilibrium position $x = 0$ is asymptotically stable if the singular point $x = 0$ of the system (Γ) is asymptotically stable.

Theorem 3. The fixed point $x = 0$ of the transformation given by (3.10) and the corresponding position of equilibrium $x = 0$ are unstable when-

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* Theorem 2 is an extension of certain considerations of Section 3 of the paper by Neimark [8], which were employed, for relay systems, in [9].

ever at least one of the eigenvalues of the matrix A has a positive real part.

Proof of Theorem 2. If all Re $\lambda(A) < 0$, then there exists a positive-definite quadratic form $V(x, x)$, for which, in view of the linear system of equations $dx/dt = Ax$, the derivative $dV/dt = 2V(x, Ax)$ is a negative-definite quadratic form. Here we have denoted by $V(x, y)$ the bilinear form which corresponds to the quadratic form $V(x, x)$. Then, for any vector x

$$V(x,\ Ax) \leqslant -rV(x,\ x) \qquad (r > 0)$$

Let $0 < \epsilon \leqslant \epsilon^*$. Then

$$\begin{gathered} V[x + \varepsilon(Ax + z),\ x + \varepsilon(Ax + z)] \leqslant \\ \leqslant V(x,\ x) + 2\varepsilon V(x,\ Ax + z) + \varepsilon\varepsilon^* V(Ax + z,\ Ax + z) = \\ = V(x,\ x) + 2\varepsilon\Big\{V(x,\ Ax) + \tfrac{1}{2}\varepsilon^* V(Ax,\ Ax) + V(x, z) + \varepsilon^* V(Ax, z) + \\ + \tfrac{1}{2}\varepsilon^* V(z,\ z)\Big\} \end{gathered} \tag{3.12}$$

Choose the number $\epsilon^* > 0$ so small that the following inequality holds:

$$V(x,\ Ax) + \tfrac{1}{2}\varepsilon^* V(Ax,\ Ax) \leqslant -(r - \eta)V(x,\ x)$$

where $\eta > 0$ is a sufficiently small number, smaller than $1/2\ r$. Further, choose the number $\Delta > 0$ so small that, for $V(x, x) < \Delta$ we have:

(1) the inequality $|x| < \epsilon^*$,

(2) the expression in the curly brackets in (3.12) is less than or equal to

$$-(r - 2\eta)V(x,\ x)$$

(3) the inequality $x_1 \geqslant 0$ implies the inequality $x_1^{(1)} \geqslant 0$.

Then, setting $q = r - 2\eta > 0$ from (3.12) it follows that

$$\begin{gathered} V[x + \varepsilon(Ax + z),\ x + \varepsilon(Ax + z)] \leqslant (1 - 2q\varepsilon)V(x,\ x) \\ \text{при } V(x,\ x) < \Delta,\ 0 \leqslant \varepsilon \leqslant \varepsilon^* \end{gathered} \tag{3.13}$$

Replacing ϵ by $x_1 > 0$ in (3.13), and noting that $x + x_1(Ax + z) = x^{(1)}$ we obtain

$$V(x^{(1)},\ x^{(1)}) \leqslant (1 - 2qx_1)V(x,\ x)$$

and, more generally

$$V(x^{(m)},\ x^{(m)}) \leqslant V(x,\ x) \prod_{k=0}^{m-1} (1 - 2qx_1^{(k)}) \qquad (3.14)$$

From this it follows that $V(x^{(m)},\ x^{(m)}) < V(x,\ x) < \Delta$, and thus $|x^{(m)}| < \epsilon^*$, $x_1^{(m)} \geqslant 0$ for sufficiently small $\Delta > 0$. Thus it has been shown that $x = 0$ is a stable fixed point of the mapping G.

According to the lemma proved above, $x_1^{(0)} + x_1^{(1)} + x_2^{(2)} + \ldots = \infty$ for $x_1 > 0$. However,

$$\prod_{k=0}^{\infty} (1 - 2qx_1^{(k)}) = 0$$

and, letting $m \to \infty$, from (3.14) we deduce that

$$\lim_{m\to\infty} V(x^{(m)},\ x^{(m)}) = 0, \quad \text{or} \quad |x^{(m)}| \to 0 \text{ for } m \to \infty \text{ and } x_1 > 0$$

The proof of Theorem 2 is complete.

Proof of Theorem 3. If at least once, we have Re $\lambda(A) > 0$, then there exists a quadratic form $V(x,\ x)$, which takes on positive values at some points, satisfying

$$V(x,\ Ax) \geqslant \mu V(x,\ x) + r|x|^2 \qquad (\mu > 0,\ r > 0) \qquad (3.15)$$

We consider two different cases.

First case. Suppose that $V(Ax + z,\ Ax + z) < 0$. In this case, one obtains the inequalities (3.12), but with the sign $<$ replaced by $>$. Proceeding analogously to the proof of Theorem 2, we obtain, for $0 \leqslant \epsilon < \epsilon^*$, $|x| < \Delta < \epsilon^*$, where ϵ^* and Δ are sufficiently small:

$$V[x + \varepsilon(Ax + z),\ x + \varepsilon(Ax + z)] \geqslant (1 + 2\mu\varepsilon)V(x,\ x) + \varepsilon q|x|^2 \qquad (3.16)$$

where q is any positive number such that $q < 2r$.

Second case. Suppose that $V(Ax + z,\ Ax + z) > 0$. The relation (3.16) holds, in view of (3.15), since in the present case

$$V[x + \varepsilon(Ax + z),\ x + \varepsilon(Ax + z)] > V(x,\ x) + 2\varepsilon V(x,\ Ax + z)$$

Choose x such that $V(x,\ x) > 0$ and $x_1 > 0$. Then, from (3.16) we obtain

$$V[x + \varepsilon(Ax + z),\ x + \varepsilon(Ax + z)] > V(x,\ x) + \varepsilon q|x|^2 \qquad (3.17)$$

Replacing ϵ by x_1 in (3.17), we obtain

$$V(x^{(1)}, x^{(1)}) > V(x, x) + qx_1|x|^2 \tag{3.18}$$

Let

$$M = \max_{x \neq 0} \frac{V(x, x)}{|x|^2}$$

Then $M > 0$, and (3.18) yields

$$V(x^{(1)}, x^{(1)}) > (1 + hx_1) V(x, x) \qquad \left(h = \frac{q}{M} > 0\right) \tag{3.19}$$

Let us suppose that the fixed point $x = 0$ of the transformation G, mapping the half-space $x_1 \geqslant 0$ into itself, is stable. Then there exists a sufficiently small number $\delta > 0$ such that, for $|x| < \delta$ and $x_1 > 0$ all $|x^{(k)}| < \Delta$ and all $x_1^{(k)} > 0$. Then

$$V(x^{(m)}, x^{(m)}) \geqslant V(x, x) \prod_{k=0}^{m+1} (1 + hx_1^{(k)}) \tag{3.20}$$

But, according to the lemma, $x_1^{(0)} + x_1^{(1)} + x_1^{(2)} + \ldots = \infty$, and hence

$$\prod_{k=0}^{\infty} (1 + hx_1^{(k)}) = \infty$$

From (3.20), since $V(x, x) > 0$, we have

$$\lim V(x^{(m)}, x^{(m)}) = \infty \qquad \text{for } m \to \infty$$

which contradicts the assumption that the fixed point $x = 0$ of the transformation G is stable. The theorem is proved.

Theorems 2 and 3 give sufficient conditions for the stability and the instability of the position of equilibrium $x = 0$ in the case when the vectors $f^+(0)$ and $f^-(0)$ are colinear. The elements of the matrix A which appears in these theorems may be immediately expressed in terms of the coefficients of the power-series expansions of the functions $f_j^+(x)$ $f_j^-(x)$:

$$f_j^{\pm}(x) = c_j^{\pm} + \sum_{k=1}^{n} c_{jk}^{\pm} x_k + \sum_{k,l=1}^{n} c_{jkl}^{\pm} x_k x_l + \ldots$$

$$(j = 1, \ldots, n;\ c_{jkl}^{\pm} = c_{jlk}^{\pm})$$

by means of the formulas

$$a_{11} = -\left[\frac{1}{3c_1^2c_{n1}}\left(c_1c_{n1}c_{11} + c_1c_{n1}c_{nn} - c_1^2c_{n1} - 2c_1\sum_{k=2}^{n-1} c_kc_{n1k} + 4c_{n1}\sum_{k=2}^{n-1} c_kc_{1k}\right)\right]_-^+$$

$$a_{jk} = \left[\frac{2}{c_1^2}(c_jc_{1k} - c_1c_{jk})\right]_-^+ \qquad (j, k = 2, 3, \ldots, n-1)$$

where we have used the notations

$$[\Phi(c)]_-^+ = \Phi(c^+) - \Phi(c^-)$$

The expression for the elements a_{k1} with $k > 1$ is not given, since these elements have no influence on the eigenvalues of the matrix.

The formulas given are obtained after the computation of the second-order terms on the right-hand sides of the equations defining the transformations G_1, G_2, and G.

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